Permutations and Combinations

Introduction

Would you like to win the lottery or the football pools or the sweepstake raffle? The big question is, why don't you win these things so easily? What's to stop you from choosing your six numbers and winning the lottery one Saturday evening and then going on your dream vacation?

Well, there's nothing to stop us winning anything. After all, all we have to do is to choose the correct six numbers! The most important point to remember is that in order to win the lottery we have to find EXACTLY the right combination of six numbers that will win us the millions we dream of.

We will soon understand why, if we have to pick 3 horses from the field of 20 entered in the Kentucky Derby to finish 1st 2nd and 3rd, then there are 6,840 ways to choose from. And if we have to select 6 numbers from 45 on a scratch off lottery ticket, there are, in fact, 8,145,060 ways of doing that. If we have to choose 6 numbers and there are 49 numbers to choose from on the lottery ticket, there are 13,983,816 ways to do that. And if we want to win the NCAA Basketball pool, where we need to predict the “Elite” 8 finalists drawn from 64 teams entered, we can choose 8 teams from 64 entered 4,426,165,368 different ways

We can start to see now why it's difficult to win the lottery, the football pools and the sweepstake!

Permutations and Combinations

So far we have talked generally about permutations and combinations and we have seen some huge numbers developing out of much smaller numbers. Let's shrink the numbers now, then, and talk about permutations first and then combinations to see exactly what it is that we are talking about and trying to achieve.

Permutations

A permutation is defined as being an ordered arrangement or grouping of a set of numbers, items etc and any one of a range of possible groupings, according to the Concise Oxford Dictionary. Oakshott adds that in the case **of permutations, the order of selection is important.**

Let's take a simple example of what a permutation really means:

Imagine that we are choosing two subjects to study out of a total of three different subject: Math, English and Science. We want to know how many ways in which we can write our daily schedule (i.e. order of taking the classes)

Permutations
1 Math English
2 Math Science
3 English Math
4 English Science
5 Science Math
6 Science English

We find a total of six permutations of our three subjects if we want to choose them in groups of two.

Next imagine now that we have been given greater choice: now we have four subjects to choose from. We want to know the number of ways we can write our schedule choosing two subjects from four: Math, English, Science and Art. Here's what is now possible.

Permutations
1 Math English
2 Math Science
3 Math Art
4 English Math
5 English Science
6 English Art
7 Science Math
8 Science English
9 Science Art
10 Art Math
11 Art English
12 Art Science

… a total of 12 permutations or ways in which we can put together a schedule of two subjects from the four available.

For you to do: Imagine now that French is added to the list of subjects available for you to choose from. How many ways are there now in which you can choose a schedule of two subjects from the five on offer?

Did you get 20 different ways? Good!

Unrealistic Method! What we have seen so far is fine but the method we have used isn't exactly realistic is it? For example, if you are about to choose your options at school or university and you have 15 subjects to choose any three from and you are interested to find out how many ways in which you might choose a schedule your three subjects, you wouldn't want to make a list of them like we've been doing so far: after all, there are 2,730 permutations of 3 from 15 … 32,760 when we are permuting 4 from 15 … and so on.

One way of dealing with this problem though is to imagine that you have, say, three numbers, 1, 2 and 3 to arrange. How many ways are there to arrange them? This is how many:

Digit 1	Digit 2	Digit 3
3 numbers to choose from	2 numbers to choose from	1 number to choose from

That is, for our first number choice, we can choose from all three available numbers; for the second number we only have two numbers to choose from and for the final number ... well, there's only one left.

Hence the number of ways of arranging 1, 2 and 3 is 3 * 2 * 1 = 6

Try this now: you have the letters A, B, C and D to arrange in a group. You want to know how many different ways you can arrange them. Here's the solution to that problem.

Letter 1	Letter 2	Letter 3	Letter 4
4 letters to choose	3 letters to choose	2 letters to choose	1 letter to choose

That is, for the first letter, you can choose from all four letters, for the second placed letter, you can only choose from the three that are left; and so on. The number of ways for organizing these four letters, then, is 4 * 3 * 2 * 1 = 24.

Try it; see if you can find all 24 ways!

Mathematicians to the Rescue!

As ever, the world's mathematicians have done all the necessary work for us, and they have developed a relatively simple formula for us to work with as we seek answers to these permutation problems.

Permutation Formula: nPr = n!/(n-r)!

Where n is the group size,
r is the number of items to be selected, without replacement, from the group.

n! Is read as n factorial and where, for example, n is 5 it means

n! = 5 x 4 x 3 x 2 x 1

(n - r)! Means the factorial of the result of having subtracted r from n.

Reworking our examples we can use this formula now, starting with the smallest and easiest numbers:

Subjects: perm 2 from 3, 2 from 4, 2 from 5 then 3 from 15

2 from 3 subjects: 3!/((3-2)!) = 6

2 from 4 subjects: 4!/((4-2)!) = 12

2 from 5 subjects: 5!/((5-2)!) = 20

3 from 15 subjects: 15!/((15-3)!) = 2730

Further Examples

1 If you want to think about permutations a little more, you can evaluate the number of possible PIN numbers that a bank can assign to a credit or debit card. Calculate the possible number of PIN numbers that can be generated when the bank uses 4 digits for a PIN. Assume for this calculation, as for any permutation calculation, that once you have used a number it cannot be used again.

Did you get 5040?

2 The “combination” lock on the safe at the local Sportsmen's Club has 100 numbers around it and the combination itself is four digits long. I. Crackem is a burglar and he knows all about the safe … how many different permutations would he have to take into account if he were to guarantee to be able to open the safe?

Did you get 94,109,400? If he takes 5 seconds for each try, how long might it take? (ans. Less than 15 years)

As a matter of interest, Mr. Crackem, you need to know the length of the code before you can even begin to crack it. In other words, if the Club's safe had a 3 from 100 code, you know that there are 970,200 different arrangements of the numbers … consider a code that is 4 from 100 (94,109,400) or 5 from 100 (9,034,502,400) … don't give up your day job and forget sitting in the vaults with your stethoscope glued to the door of the safe waiting for all those clicks, it just isn't going to happen!

Remember the definition of permutations that said

A permutation is defined as being an ordered arrangement or grouping of a set of numbers(i.e., the order of selection is important.)

A numerical example will show why the ordering of numbers in a permutation is important. Let's imagine that the Sportsmen's Club's safe has only 3 numbers around it: 1, 2 and 3; and it has a 2-digit code, this gives us the following possibilities:

Permutations of simple safe "codes".

#**1.** 1-2
#**2.** 1-3
#**3.** 2-1
#**4.** 2-3
#**5**. 3-1
#**6**. 3-2

Relate this list to the picking two from three subjects example. If the subjects were numbered 1,2, and 3,and ***if you just had to pick two subjects from three to take this semester*** <u>**without**</u> ***regard to which one you took first in the day***, we would say that in the above list of permutations #1 and #3 were equivalent to each other since they both have the subjects 1 and 2 in them. Similarly, we would say that permutations 2 and 5, and 4 and 6 are equivalent since they have the same subject pairs in them.

However, here we can see that if we tried 1 2 instead of 2 1 and 2 1 is the correct code, the safe would not open … this is an example of why we say that ordering is important with permutations.

The ordering aspect, therefore, is a vital point that we need to explore a little more; and that takes us on to combinations.

Combinations

To reiterate, what if you just had to pick 2 subjects from 4 to take this semester <u>**without**</u> ***regard to which one you will take first in the day? Then in our previous list off possible schedules(reproduced below), we can see some pairs are listed more than once since order is not a consideration!***

1 Math English
2 Math Science
3 English Math
4 English Science
5 Science Math
6 Science English

Again, notice here that 1 is really the same as 3; 2 is the same as 5; and 4 is the same as 6. when order is not a concern. That is, this example shows us that when we derive the permutations from lists of objects, digits, subjects, or variables etc… , we generate repetitions of subsets with different ordering(permutations).

The Concise Oxford Dictionary defines a combination as a group of things chosen from a larger number **without** regard to their order. In other words, combinations do not count the number of possible codes that open safes. They are concerned with such problems as the ways of choosing subjects from a given list when the order does not matter, or from working out the possible results from betting on the tossing of a number of coins and so on.

Let's go back to the choice of two from three subjects example we have already seen. And compare the difference if we just needed to pick the class titles ---

i.e. Permutations, pick and make a schedule(order them)

Combinations, just pick the classes you wish to take this semester

	Permutations	**Combinations**
1	math English	math English
2	math science	math science
3	English math	
4	English science	English science
5	science math	
6	science English	
C	**6**	**3**

and for the two from four subjects we have:

	Permutations	**Combinations**
1	math English	math English
2	math science	math science
3	math art	math art
4	English math	
5	English science	English science
6	English art	English art
7	science math	
8	science English	
9	science art	science art
10	art math	
11	art English	
12	art science	
C	**12**	**6**

For You to Do: work down the table for the two from five subject now … how many subjects can be chosen by permutations and by combinations?

Did you get 20 and 10 respectively?

Have you spotted that the combinations results return only UNIQUE sets of values? That is, with combinations, we can only choose math and English but not also English and math; English and science but not also science and English.

Mathematicians have devised a formula to deal with these situations:

Combination Formula: $_nC_r = n!/[r!(n-r)!]$

where n is group size
n! is n factorial
r is the items to be selected
r! is r factorial

Let's apply this to the two subject from three, two from four subjects and then three from 15 subjects:

$$2 \text{ subjects from } 3 = {}^3C_2 \frac{3!}{2!(3-2)!} = 3$$

$$2 \text{ subjects from } 4 = {}^4C_2 \frac{4!}{2!(4-2)!} = 6$$

$$3 \text{ subjects from } 15 = {}^{15}C_3 \frac{15!}{3!(15-3)!} = 455$$

We see that there is a pattern to the way that combinations stem from permutations: We just divide the number of permutations by the number of permutations of the number of the subset chosen, that is divide by r! The following may help you see this if you use your calculator to compare say,

$_6P_4/4!$ with $_6C_4$ etc..

Now go back to the introduction and see if you agree with those large numbers. Then see if you can find other applications for permutation and or combination theory. (e.g. If a jar contains 20 green,20 orange and 20 yellow M&Ms, and you reach in and grab 5, then what is the probability they are all the same color?